[This question paper contains 6 printed pages.]

(16) Your Roll No. 2022

Sr. No. of Question Paper : 1114 A

Unique Paper Code : 32351601

Name of the Paper : BMATH 613 – Complex Analysis

Name of the Course : **B.Sc. (H) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. 3. **All** questions are compulsory.

3. Attempt **two** parts from each question.

1. (a) Sketch the region onto which the sector $r \leq 1$, $0 \leq \theta \leq \pi$ is mapped by the transformation $w = z^2$ and $w = z^3$. (6)

   (b) (i) Find the limit of the function $f(z) = \frac{(\bar{z})^2}{z}$ as z tends to 0.

P.T.O.

(ii) Show that $\lim_{z \to 1+\sqrt{3}i} \frac{z^2-2z+4}{z-1-\sqrt{3}i} = 2\sqrt{3}i$. (3+3=6)

(c) Let u and v denote the real and imaginary components of the function f defined by means of the equations

$$f(z) = \begin{cases} \bar{z}^2/z & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$$

Verify that the Cauchy-Riemann equations are satisfied at the origin z = (0,0). (6)

(d) If $\lim_{z \to z_0} f(z) = F$ and $\lim_{z \to z_0} g(z) = G$, prove that

$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{F}{G}$ if $G \neq 0$. (6)

2. (a) Find the values of z such that

(i) $e^z = 1 + \sqrt{3}i$, (ii) $e^{(2z-1)} = 1$. (3.5+3=6.5)

(b) Show that the roots of the equation $\cos z = 2$ are $z = 2n\pi + i\cosh^{-1}2$ $(n = 0, \pm 1, \pm 2, \ldots)$, Then express them in the form $z = 2n\pi \pm i\ln(2+\sqrt{3})$ $(n = 0, \pm 1, \pm 2, \ldots)$. (3.5+3=6.5)

(c) Show that (3.5+3=6.5)

(i) $\log(1+i)^2 = 2Log(1+i)$,

(ii) $\log(-1+\sqrt{3}i) = \ln 2 + 2\left(n+\frac{1}{3}\right)\pi i \quad (n = 0, \pm 1, \pm 2, \ldots)$

(d) Show that $\overline{\exp(iz)} = \exp(i\bar{z})$ if and only if $z = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. (6.5)

3. (a) (i) State mean value theorem of integrals. Does it hold true for complex valued functions? Justify.

(ii) Evaluate $\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta$. (3+3=6)

(b) Parametrize the curves C_1 and C_2, where

C_1: Semicircular path from -1 to 1

C_2: Polygonal path from the vertices $-1, -1+i, 1+i$ and 1

Evaluate $\int_{C_1} z\, dz$ and $\int_{C_2} z\, dz$. (3+3=6)

(c) For an arbitrary smooth curve $C: z = z(t)$, $a \le t \le b$, from a fixed point z_1 to another fixed point z_2, show that the value of the integrals

(i) $\int_{z_1}^{z_2} z\, dz$ and

(ii) $\int_{z_1}^{z_2} dz$

depend only on the end points of C. (3+3=6)

(d) State ML inequality theorem. Use it to prove that $\left|\int_C \frac{dz}{z^4}\right| \le 4\sqrt{2}$, where C denotes the line segment from z = i to z = 1. (2+4=6)

4. (a) A function f(z) is continuous on a domain D such that all the integrals of f(z) around closed contours lying entirely in D have the value zero. Prove that f(z) has an antiderivative throughout D. (6.5)

(b) State Cauchy Goursat theorem. Use it to evaluate the integrals

(i) $\int_C \frac{1}{z^2+2z+2} dz$, where C is the unit circle $|z| = 1$

(ii) $\int_C \frac{2z}{z^2+2} dz$, where C is the circle $|z| = 2$

(2.5+2+2=6.5)

(c) State and prove Cauchy Integral Formula.

(2+4.5=6.5)

(d) (i) State Liouville's theorem. Is the function f(z) = cos z bounded? Justify.

(ii) Is it true that 'If p(z) is a polynomial in z then the function f(z) = 1/p(z) can never be an entire function'? Justify (4.5+2=6.5)

5. (a) If a series $\sum_{n=1}^{\infty} z_n$ of complex numbers converges then prove $\lim_{n\to\infty} z_n = 0$. Is the converse true? Justify. (6.5)

(b) Find the integral of $\int_C \frac{\cosh \pi z}{z^3+z}$ where C is the positively oriented circle |z| = 2. (6.5)

(c) Find the Taylor series representation for the function $f(z) = \frac{1}{z}$ about the point $z_0 = 2$. Hence prove that $\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n$ for $|z-2| < 2$. (6.5)

(d) If a series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges to f(z) at all points interior to some circle $|z - z_0| = R$, then prove that it is the Taylor series for the function f(z) in power of $z - z_0$. (6.5)

6. (a) For the given function $f(z) = \frac{z+1}{z^2+9}$ find the poles, order of poles and their corresponding residue. (6)

(b) Write the two Laurent Series in powers of z that represent the function $f(z) = \frac{1}{z+z^3}$ in certain domains and specify those domains. (6)

(c) Suppose that $z_n = x_n + iy_n$, ($n = 1,2,3,...$) and $S = X + iY$. Then prove that

$\sum_{n=1}^{\infty} z_n = S$ iff $\sum_{n=1}^{\infty} x_n = X$ and $\sum_{n=1}^{\infty} y_n = Y$. (6)

(d) Define residue at infinity for a function f(z). If a function f(z) is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then prove that

$$\operatorname*{Res}_{z=\infty} f(z) = -\operatorname*{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$ (6)

[This question paper contains 10 printed pages.]

(17) Your Roll No. 2022

Sr. No. of Question Paper : 1211 A

Unique Paper Code : 32357614

Name of the Paper : DSE-3 MATHEMATICAL FINANCE

Name of the Course : B.Sc. (Hons) Mathematics CBCS (LOCF)

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory and carry equal marks.
4. Use of Scientific calculator, Basic calculator and Normal distribution tables all are allowed.

1. (a) Explain Duration of a zero-coupon bond. A 4-year bond with a yield of 10% (continuously compounded) pays a 9% coupon at the end of each year.

(i) What is the bond's price?

(ii) Use duration to calculate the effect on the bond's price of a 0.3% decrease in its yield?

(You can use the exponential values: $e^x = 0.9048$, 0.8187, 0.7408, and 0.6703 for x = –0.1, –0.2, –0.3, and –0.4, respectively)

(b) Explain Continuous Compounding. Suppose R_c denotes rate of interest with continuous compounding and R_m denotes equivalent rate with compounding m times per annum. Find the relation between R_c and R_m.

(c) An investor receives ₹ 1100 in one year in return for an investment of ₹ 1000 now. Calculate the percentage return per annum with :

(i) Annual compounding

(ii) Semi-annual compounding

(iii) Continuous compounding.

(You can use: ln(1.1) = 0.953)

(d) Define Bond Yield and Par Yield. Suppose that the 6-month, 12-month, 18-month, and 24-month zero rates are 5%, 6%, 6.5% and 7% respectively. What is the 2-year par yield? (You can use the exponential values: $e^x = 0.9753, 0.9418, 0.9071, 0.8694$ for $x = -0.025, -0.06, -0.0975, -0.14$, respectively.)

2. (a) Explain Hedging. A United States company expects to pay 1 million Canadian dollars in 6 months. Explain how the exchange rate risk can be hedged using

(i) A Forward Contract

(ii) An Option.

(b) (i) What is the difference between the over-the counter market and the exchange-traded market?

(ii) An investor enters a short forward contract to sell 175,000 British pounds for US dollars at an exchange rate of 1.900 US dollars per pound. How much does the investor gain or lose if the exchange rate at the end of the contract is 2.420?

(c) A 1-year forward contract on a non-dividend paying stock is entered into when the stock price is ₹ 40, and the risk-free rate of interest is 10% per annum with continuous compounding. What is the forward price? Justify using no arbitrage arguments. ($e^{0.1} = 1.1052$)

(d) (i) A trader writes an October call option with a strike price of ₹ 35. The price of the option is ₹ 6. Under what circumstances does the trader make a gain,

(ii) Suppose that you own 6,000 shares worth ₹ 75 each. How can put options be used to provide an insurance against a decline in the value of the holding over the next 4 months?

3. (a) Draw the diagrams illustrating the effect of changes in stock price, strike price, and expiration date on European call and put option prices when

$S_0 = 50$, $K = 50$, $r = 5\%$, $\sigma = 30\%$, and $T = 1$.

(b) Derive the put-call parity for European options on a non-dividend-paying stock. Use put-call parity to derive the relationship between the delta of a European call and the delta of a European put on a non-dividend-paying stock.

(c) An investor sells a European call on a share for ₹ 4. The stock price is ₹ 47 and the strike price is ₹ 50. Under what circumstances does the investor make a profit? Under what circumstances will the option be exercised? Draw a diagram showing the variation of the investor's profit with the stock price at the maturity of the option.

(d) Define upper bound and lower bound for European options on a non-dividend- paying stock. What is a lower bound for the price of a 3-month European put option on a non-dividend-paying stock when the stock price is ₹38, the strike price is ₹40, and the risk-free interest rate is 10% per annum? Justify using no arbitrage arguments. ($e^{-0.04}$ = 0.9753)

4. (a) A 4-month European call option on a dividend-paying stock is currently selling for ₹ 50. The stock

price is ₹ 640, the strike price is ₹ 600, and a dividend of ₹ 8 is expected in 1 month. The risk-free interest rate is 12% per annum for all maturities. What opportunities are there for an arbitrageur? ($e^{-0.04} = 0.9608$)

(b) Consider a one-period binomial model where the stock can either go up from S_0 to S_0u $(u > 1)$ or down from S_0 to S_0d $(d < 1)$. Suppose we have an option with payoff f_u if the stock moves up and payoff f_d if the stock moves down. By considering a portfolio consisting of long position in Δ shares of stock and a short position in the option, find the price of the option. Explain how the price can be expressed as an expected payoff discounted by the risk-free interest rate.

(c) A stock price is currently ₹ 50. It is known that at the end of two months it will be either ₹ 53 or ₹ 48. The risk-free interest rate is 12% per annum with continuous compounding. What is the value of a two-month European call option with a strike price of ₹ 49? Use no-arbitrage arguments. ($e^{0.02} = 1.0202$)

(d) Consider a two-period binomial model with current stock price S_0 = ₹ 100, the up factor u = 1.3, the down factor d = 0.8, T = 1 year and each period being of length six months. The risk-free interest rate is 5% per annum with continuous compounding. Construct the two-period binomial tree for the stock. Find the price of an American put option with strike K = ₹ 95 and maturity T = 1 year. ($e^{-0.025}$ = 0.9753)

5. (a) Stock price in the Black-Scholes model satisfies

$$\ln S_T \sim \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

where $\phi(m, v)$ denotes a normal distribution with mean m and variance v. Find $Var[S_T]$.

(b) What is the price of a European put option on a non-dividend-paying stock when the stock price is ₹ 69, the strike price is ₹ 70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

(You can use exponential values: $e^{-0.0144}$ = 0.9857, $e^{-0.025}$ = 0.9753)

(c) Let V be a lognormal random variable with ω being the standard deviation of ln V. Prove that

$$E[\max(V - K, 0)] = E(V)N(d_1) - KN(d_2)$$

where

$$d_1 = \frac{\ln\left[\frac{E(V)}{K}\right] + \frac{\omega^2}{2}}{\omega}, \quad d_2 = \frac{\ln\left[\frac{E(V)}{K}\right] - \frac{\omega^2}{2}}{\omega}$$

and E denotes the expected value. Use this result to derive the Black-Scholes formula for the price of a European call option on a non-dividend paying stock.

(d) A stock price is currently ₹ 50. Assume that the expected return from the stock is 18% and its volatility is 30%. What is the probability distribution for the stock price in 2 years? Calculate the mean and standard deviation of the distribution. ($e^{0.18}$ =1.1972)

6. (a) Discuss gamma of a portfolio of options and calculate the gamma of a European call option on a non-dividend-paying stock where the stock price is ₹ 49, the strike price is ₹ 50, the risk-free

interest rate is 5% per annum and the time to maturity is 20 weeks, and the stock price volatility is 30% per annum. (ln(49/50) = −0.0202)

(b) What is the relationship between delta, theta and gamma of an option? Show by substituting for various terms in this relationship that it is true for a single European put option on a non-dividend-paying stock.

(c) Find the payoff from a bear spread created using put options. Also draw the profit diagram corresponding to this trading strategy.

(d) Companies X wishes to borrow US dollars at a fixed interest rate. Company Y wishes to borrow Indians rupees at a fixed rate of interest. The amounts required by the two companies are roughly the same at the current exchange rate. The companies have been quoted the following interest rates, which have been adjusted for the impact of taxes :

	Rupees	Dollars
Company X	9.6%	6.0%
Company Y	11.1%	6.4%

Design a swap that will net a bank, acting as intermediary, 50 basis points per annum. Make the swap equally attractive to the two companies and ensure that all foreign exchange risk is assumed by the bank.

[This question paper contains 8 printed pages.]

(18) Your Roll No. 2022

Sr. No. of Question Paper : 1212 A

Unique Paper Code : 32357615

Name of the Paper : Introduction to Information Theory and Coding

Name of the Course : **B.Sc. (H) Mathematics**

Semester : VI

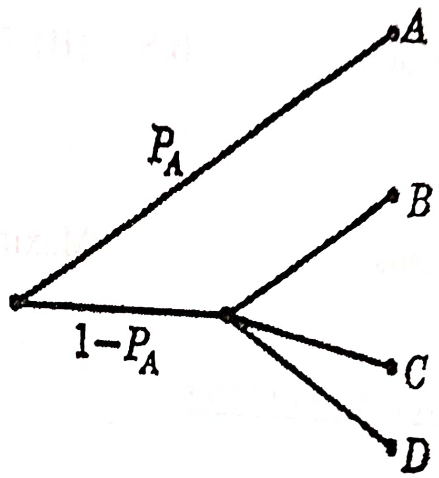
Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.
4. **All** questions carry equal marks.

1. (a) Define entropy of a binary source. Calculate the entropy of a source with an alphabet having 64 symbols, where each symbol is equally probable.

(b) Verify the additive rule of entropies for the probability scheme, where the input probabilities are $P(A)=\frac{1}{2}$, $P(B)=\frac{1}{4}$, $P(C)=\frac{1}{8}$, $P(D)=\frac{1}{8}$.



(c) A transmitter has an alphabet consisting of five letters $\{x_1, x_2, x_3, x_4, x_5]$ and the receiver has an alphabet of four letters $\{y_1, y_2, y_3, y_4\}$. The joint probabilities are given as :

	y_1	y_2	y_3	y_4
x_1	0.25	0	0	0
x_2	.15	.25	0	0
x_3	0	0.05	0.10	0
x_4	0	0	0.05	0.10
x_5	0	0	0.05	0

Compute all the entropies for this channel.

(d) Discuss the properties of the entropy function.

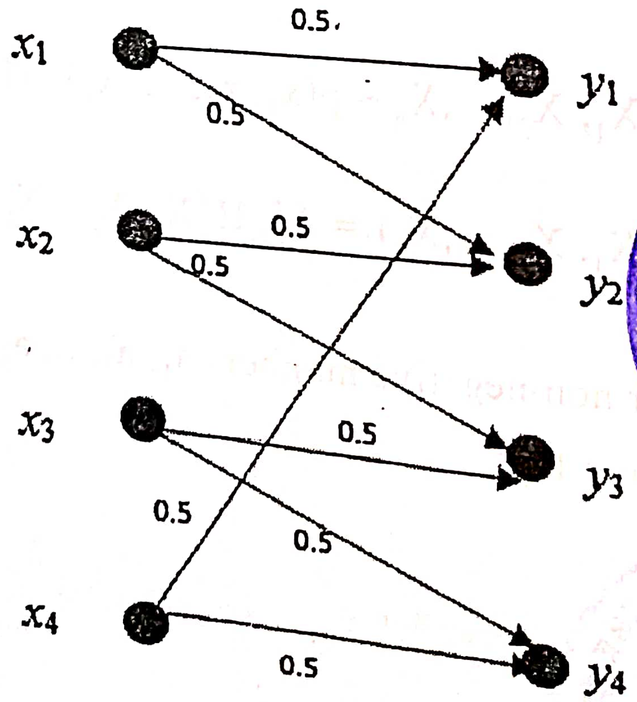
2. (a) What is a channel? Compute the capacity for the channels having symmetric noise structures.

(b) A discrete source emits one of five symbols once every milliseconds with probabilities 1/2, 1/4, 1/8, 1/16 and 1/16. Find the source entropy and information rate.

(c) Explain Binary erasure channel. Derive the expressions for its source entropy, conditional entropy H(X|Y) and the channel capacity.

(d) If (X, Y) are two random variables, show that H(X, Y) = H(X) + H(Y|X). Hence or otherwise prove that H(X, Y|Z) = H(X|Z) + H(Y|X,Z).

3. (a) Given below the noise characteristic



For equi-probable transmission probabilities of the messages x_1, x_2, x_3, x_4 determine the following :

(i) Receiver entropy $H(Y)$

(ii) Conditional entropy $H(Y|X)$

(iii) Mutual information $I(X ; Y)$

(b) If (X, Y) be the joint random variable of the random variables X and Y with $(X, Y) \sim p(x, y)$. Then define the mutual information $I(X, Y)$ and show that

(i) $I(X;Y) = H(Y) - H(Y|X)$

(ii) $I(X;Y) = H(X) + H(Y) - H(X, Y)$

(c) If $X_1, X_2, \ldots, X_n \sim p(x_1, x_2, \ldots, x_n)$. Then prove that

$$H(X_1, X_2, \ldots, X_n) = \Sigma_{i=1}^{n} H\left(X_i \mid X_{i-1}, X_{i-2}, \ldots, X_1\right)$$

(d) For non-negative numbers $a_1, a_2, \ldots, a_n$ and b_1, b_2, b_n prove that

$$\Sigma_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq \left(\Sigma_{i=1}^{n} a_i\right) \log \frac{\Sigma_{i=1}^{n} a_i}{\Sigma_{i=1}^{n} b_i}$$

With equality iff $\frac{a_i}{b_i}$ = Constant. Hence, prove Kullback - Leibler distance function $D(p||q)$ is convex in the pair (p,q), where p and q be probability mass functions.

4. (a) Define nearest neighbour decoder. For a repetition code of length 3 a codeword is transmitted through a BSC with crossover probability $p=0.2$, Find the probability that a received word is decoded as the transmitted word by the nearest neighbour decoder.

(b) Let C be an (n, M, d) code over F and let τ and σ be nonnegative integers such that $2\tau + \sigma \leq d-1$. Prove that there is a decoder $D: F^n \rightarrow C \cup \{\text{"e"}\}$ with the following properties :

(i) If the number of errors is τ or less, then the errors will be recovered correctly.

(ii) Otherwise, if the number of errors is $\tau + \sigma$ or less, then they will be detected.

(c) Construct the $GF(2^3)$ as a ring of residues over GF(2) modulo the polynomial $P(x) = x^3 + x + 1$.

(d) Prove that

(i) Let a(x), b(x), and c(x) be polynomials over F such that $c(x) \neq 0$ and $\gcd(a(x), c(x)) = 1$. Then, prove that $c(x) \mid a(x) \cdot b(x) \Rightarrow c(x) \mid b(x)$.

(ii) Let P(x) be an irreducible polynomial over F and let a(x) and b(x) be polynomials over F. Then, $P(x) \mid a(x) \cdot b(x) \Rightarrow P(x) \mid a(x)$ or $P(x) \mid b(x)$.

5. (a) True or False. If it is false, then give the correct statement :

(i) Every set of n-k columns of Parity Check matrix H of a linear [n,k,d] code C is linearly independent.

(ii) Every set of k columns of Generator matrix G of a linear [n,k,d] code C is linearly dependent.

(iii) The dual code of MDS code is not MDS.

(iv) The systematic generator matrix is of the form (I|A), where I is an identity matrix of order (n-k) and A is k x k matrix.

(b) Define dual code of a Linear code C. Find the Parity- Check matrix of matrix of [7,4,3] Hamming code over GF(2) and its dual.

(c) Find gcd ($x^3 + x^2 + x + 1$, $x^2 + 1$). If gcd = 1, then, find inverse of $x^3 + x^2 + x + 1$ by Euclidean Algorithm.

(d) Let C be a binary (5, 2, 3) code with generator matrix,

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

(i) Find a parity-check matrix for C.

(ii) Write out the elements of the dual code $C^{\perp}$.

6. (a) State and prove the the Gilbert-Varshamov bound.

(b) Let C be a binary (4,2) code with the generator matrix,

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

(i) Find coset leaders and their syndromes.

(ii) Use syndrome decoding to decode the received vector v = 0111.

(c) Let H be a parity – check matrix of a linear code $C \neq \{0\}$. The minimum distance of C is the largest integer d such that every set of d-1 columns in H is linearly independent.

(d) State and prove the Sphere Packing Bound.

[This question paper contains 4 printed pages.]

(19) Your Roll No. 2022

Sr. No. of Question Paper : 1298 A

Unique Paper Code : 32357610

Name of the Paper : DSE-4 (Number Theory)

Name of the Course : **CBCS (LOCF) – B.Sc. (H) (Mathematics)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Attempt any **two** parts of each question.

4. Question Nos. **1** to **3**, each part carries **6.5** marks and Question Nos. **4** to **6**, each part carries **6** marks.

1. (a) Determine all solutions in the integers of the Diophantine equation $24x + 138y = 18$.

(b) A farmer purchased 100 head of livestock for a total cost of Rs. 4000. Prices were as follow: calves, Rs. 120 each; lambs, Rs. 50 each; piglets,

P.T.O.

Rs. 25 each. If the farmer obtained at least one animal of each type, how many of each did he buy?

(c) Write a short note on Goldbach conjecture.

(d) Find the remainder obtained upon dividing the sum $1! + 2! + 3! + \cdots + 100!$ by 12.

2. (a) Prove that the congruences

$x \equiv a \pmod{n}$ and $x \equiv b \pmod{m}$ admits a simultaneous solution if $\gcd(n,m) | (a-b)$; if a solution exists, confirm that it is unique modulo $\operatorname{lcm}(n,m)$.

(b) Solve the linear congruence $25x \equiv 15 \pmod{29}$.

(c) If p an q are distinct primes, prove that

$$p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}.$$

(d) State and prove Wilson's theorem.

3. (a) If f is a multiplicative function and F is defined by $F(n) = \sum_{d/n} f(d)$ then show that F is also multiplicative, explain your result when m = 8 and n = 3.

(b) Explain Mobius p- function with example and also show that

$$\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$$

for each positive integer n.

(c) Use the fact that each prime p has a primitive root to give a different proof of Wilson's theorem.

(d) Let r be a primitive root of the integer n. Prove that r^k is a primitive root of n if and only if gcd $(k, \Phi(n)) = 1$.

4. (a) Define number – theoretic function and also show that number – theoretic functions σ and τ both are multiplicative functions.

(b) Write a short note on Mobius function and show this function is multiplicative function.

(c) If p is a prime and $k > 0$, then prove $\phi(p^k) = p^k - p^{k-1}$. Explain your result by an example.

(d) Use Euler's theorem for any odd integer a, to prove $a^{33} \equiv a \pmod{4080}$.

5. (a) Find a primitive root for any integer of the form 17^k.

(b) Let p be an odd prime and gcd(a, p) = 1. Then prove that 'a' is a quadratic residue of p if and only if $a^{(p-1)/2} \equiv 1 \pmod{p}$.

(c) Let p be an odd prime and let a and b be integers that are relatively prime to p. Show that (ab/p) = (a/p) (b/p).

(d) Find the value of Legendre symbols (18/43) and (−72/131).

6. (a) Use Gauss lemma to compute Legendre symbol (5/19).

(b) Show that 7 and 18 are the only incongruent solutions of $x^2 \equiv -1 \pmod{5^2}$.

(c) Using the linear cipher $C \equiv 5P + 11 \pmod{26}$ encrypt the message CRYPTOGRAPHY.

(d) Use the Hill's cipher

$$C_1 \equiv 5P_1 + 2P_2 \pmod{26}$$

$$C_2 \equiv 3P_1 + 4P_2 \pmod{26}$$

to encrypt the message GIVE THEM TIME.

[This question paper contains 8 printed pages.]

(20) Your Roll No. 2022

Sr. No. of Question Paper : 1299 A

Unique Paper Code : 32357616

Name of the Paper : DSE-4 Linear Programming and Applications

Name of the Course : **CBCS (LOCF) – B.Sc. (H) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt any **two** parts from each question.

3. **All** questions carry equal marks.

1. (a) Solve the following Linear Programming Problem by Graphical Method :

Minimize $3x + 2y$

subject to $5x + y \geq 10$

$x + y \geq 6$

$x + 4y \geq 12$

$x \geq 0, y \geq 0$.

(b) Define a Convex Set. Show that the set S defined as :

$$S = \{(x,y) \mid x^2 + y^2 \leq 4\}$$ is a Convex Set.

(c) Find all basic feasible solutions of the equations:

$$x_1 + x_2 + 2x_3 + 3x_4 = 12$$

$$x_2 + 2x_3 + x_4 = 8$$

(d) Prove that to every basic feasible solution of the Linear Programming Problem:

$$\text{Minimize } z = cx$$

$$\text{subject to } Ax = b,\ x \geq 0$$

there corresponds an extreme point of the feasible region.

2. (a) Let us consider the following Linear Programming Problem:

$$\text{Minimize } z = cx$$

$$\text{subject to } Ax = b,\ x \geq 0$$

Let $(x_B, 0)$ be a basic feasible solution corresponding to a basis B having an a_j with $z_j - c_j > 0$ and all corresponding enteries $y_{ij} \leq 0$, then show that Linear Programming Problem has an unbounded solution.

(b) Let $x_1 = 2$, $x_2 = 1$, $x_3 = 1$ be a feasible solution to the system of equations:

$$x_1 + 4x_2 - x_3 = 5$$

$$2x_1 + 3x_2 + x_3 = 18$$

Is this a basic feasible solution? If not, reduce it to two different basic feasible solutions.

(c) Using Simplex method, find the solution of the following Linear Programming Problem:

$$\begin{aligned} \text{Minimize} \quad & x_1 - 3x_2 + 2x_3 \\ \text{subject to} \quad & 3x_1 - x_2 + 2x_3 \leq 7 \\ & 2x_1 - 4x_2 \geq -12 \\ & -4x_1 + 3x_2 + 8x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

(d) Solve the following Linear Programming Problem by Big-M method:

$$\begin{aligned}
\text{Maximize} \quad & x_1 - 4x_2 + 3x_3 \\
\text{subject to} \quad & 2x_1 - x_2 + 5x_3 = 40 \\
& x_1 + 2x_2 - 3x_3 \geq 22 \\
& 3x_1 + x_2 + 2x_3 = 30 \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}$$

3. (a) Solve the following Linear Programming Problem by Two Phase Method :

$$\begin{aligned}
\text{Maximize} \quad & x_1 + 4x_2 + 3x_3 \\
\text{subject to} \quad & x_1 + x_2 + x_3 \geq 4 \\
& --2x_1 + 3x_2 - x_3 \leq 2 \\
& x_2 - 2x_3 \leq 1 \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}$$

(b) Find the solution of given system of equations using Simplex Method:

$$3x_1 - 2x_2 = 8$$

$$x_1 + 2x_2 = 4$$

Also find the inverse of A where $A=\begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$.

(c) Using Simplex method, find the solution of the following Linear Programming Problem :

Maximize $2x_1 + x_2$
subject to $x_1 - x_2 \le 10$

$2x_1 - x_2 \le 40$

$x_1 \ge 0,\ x_2 \ge 0.$

(d) Find the optimal solution of the Assignment Problem with the following cost matrix :

Job \ Machines	I	II	III	IV	V	VI
A	4	8	5	4	6	9
B	8	3	8	4	11	7
C	9	5	7	9	8	7
D	10	9	5	6	9	9
E	5	11	9	10	10	9
F	9	5	7	10	8	7

4. (a) Find the Dual of following Linear Programming Problem :

Minimize $x_1 + x_2 + 3x_3$
subject to

$4x_1 + 8x_2 \ge 3$

$7x_2 + 4x_3 \le 6$

$3x_1 - 2x_2 + 5x_3 = 7$

$x_1 \le 0,\ x_2 \ge 0,\ x_3$ is unrestricted.

(b) State and prove the Weak Duality Theorem. Also show that if the objective function values corresponding to feasible solutions of the Primal and Dual Problem are equal then the respective solutions are optimal for the respective Problems.

(c) Using Complementary Slackness Theorem, find optimal solutions of the following Linear Programming Problem and its Dual:

Maximize $4x_1 + 3x_2$
subject to

$$x_1 + 2x_2 \leq 2$$
$$x_1 - 2x_2 \leq 3$$
$$2x_1 + 3x_2 \leq 5$$
$$x_1 + x_2 \leq 2$$
$$3x_1 + x_2 \leq 3$$
$$x_1, x_2 \geq 0.$$

(d) For the following cost minimization Transportation Problem find initial basic feasible solutions by using North West Comer rule, Least Cost Method and Vogel's Approximation Method. Compare the three solutions (in terms of the cost):

Destination / Source	A	B	C	D	E	Supply
I	15	15	16	17	15	24
II	18	19	16	20	15	38
III	16	15	22	17	20	43
Demand	27	12	32	17	17	

5\. (a) Solve the following cost minimization Transportation Problem :

Origin \ Destinations	I	II	III	IV	Availability
A	10	11	10	13	30
B	12	12	11	10	50
C	13	11	14	18	20
Requirements	20	40	30	10	

(b) Four new machines are to be installed in a machine shop and there are five vacant places available. Each machine can be installed at to one and only one place. The cost of installation of each job on each place is given in table below. Find the Optimal Assignment. Also find which place remains vacant.

Machine \ Place	A	B	C	D	E
I	13	15	19	14	15
II	16	13	13	14	13
III	14	15	18	15	11
IV	18	12	16	12	10

(c) Define Maxmin and Minmax value for a Fair Game. Using Maxmin and Minmax Principle, find the saddle point, if exists, for the following pay-off matrix :

Player 1

$$\text{Player 2} \begin{bmatrix} 1 & 3 & 6 \\ 5 & 4 & 2 \\ 2 & 1 & 3 \end{bmatrix}.$$

(d) Convert the following Game Problem into a Linear Programming Problem for player A and player B and solve it by Simplex Method :

Player B

$$\text{Player A} \begin{bmatrix} 2 & 3 & 1 \\ 4 & 2 & 6 \end{bmatrix}.$$

[This question paper contains 6 printed pages.]

(21) Your Roll No. 2022

Sr. No. of Question Paper : 1359 A

Unique Paper Code : 32351602

Name of the Paper : BMATH614: Ring Theory and Linear Algebra II

Name of the Course : B.Sc. (Hons.) Mathematics

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt any **two** parts from each question.

1. (a) (i) If D is an Integral domain, prove that D[x] is an integral domain.

(ii) If R is a commutative ring, prove that the characteristic of R[x] is same as the characteristic of R.

(b) Let $f(x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1$ in $Z_7[x]$. Compute the product $f(x)g(x)$. Determine the quotient and the remainder upon dividing $f(x)$ by $g(x)$.

(c) Let F be a field and let $I = \{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 | a_i \in F$ and $f(1) = a_n + \ldots a_0 = 0)$. Prove that I is an Ideal of F[x] and find a generator of I.

(d) Let R[x] denote the ring of polynomials with real coefficients. Then prove that $\frac{R[x]}{\langle x^2+1 \rangle}$ is isomorphic to the ring of complex numbers.

(3+3.5,6.5,6.5,6.5)

2. (a) (i) Let F be a field and $p(x) \in F[x]$ be irreducible over F. Prove that $<p(x)>$ is a maximal ideal in F[x].

(ii) Show that, $\frac{Z_2[x]}{\langle x^3+x+1 \rangle}$ is a field with 8 elements.

(b) Determine which of the polynomials below are irreducible over Q.

(i) $3x^5 + 15x^4 - 20x^3 + 10x + 20$

(ii) $x^4 + x + 1$

(c) In integral domain $Z[\sqrt{-3}]$, prove that $1+\sqrt{-3}$ is irreducible but not prime.

(d) Define Euclidean domain. Prove that every Euclidean domain is a principal ideal domain.

(3+3,3+3,6,6)

3. (a) Let $V = P_1(R)$ and V^* denote the dual space of V. For $p(x) \in V$, define

$f_1, f_2 \in V^*$ by $f_1(p(x)) = \int_0^1 p(t)dt$ and $f_2(p(x)) = \int_0^2 p(t)dt$. Prove that $\{f_1, f_2\}$ is a basis for V^*

and find a basis for V for which it is the dual basis.

(b) Let W be a subspace of finite dimensional vector space V. Prove that

$\dim(W) + \dim(W^\circ) = \dim(V)$, where W° is annihilator of W.

(c) Let T be a linear operator on $M_{n\times n}(R)$ defined by $T(A) = A^t$. Show that ± 1 are the only eigenvalues of T. Find the eigenvectors corresponding to each eigenvalue. Also find bases for $M_{2\times 2}(R)$ consisting of eigenvectors of T.

(d) Let T be a linear operator on R^3 defined by $T(a, b, c) = (3a + b, 3b + 4c, 4c)$. Show that T is digonalizable by finding a basis for R^3 consisting of eigen vectors of T. (6.5,6.5,6.5,6.5)

4. (a) Let T be a linear operator on finite dimensional vector space V and let W be the T-cyclic subspace of V generated by a non-zero vector $v \in V$. Let $k = \dim(W)$. Then prove that $\{v, T(v), \ldots\ldots\ldots, T^{k-1}(v)\}$ is basis for W.

(b) State Cayley Hamilton Theorem. Verify the theorem for linear operator $T: R^2 \to R^2$ defined by $T(a, b) = (a + 2b, -2a + b)$.

(c) Let T be a linear operator on R^3 defined by $T(a, b, c) = (3a - b, 2b, a - b + 2c)$. Find the characteristic polynomial and minimal polynomial of T.

(d) (i) Let T be an invertible linear operator. Prove that a scalar λ is an eigen value of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

(ii) Prove that similar matrices have the same characteristic polynomial. (6,6,6,3+3)

5. (a) Show that in a complex inner product space V over field F. For x, y ∈ V, prove the following identities

(i) $\langle x, y\rangle = \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2$ if $F = R$

(ii) $\langle x, y\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \left\|x + i^k y\right\|^2$ if F = C, where

$i^2 = -1$.

(b) Let V be an inner product space, and let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal subset of V. Prove the Bessel's Inequality :

$$\|x\|^2 \geq \sum_{i=1}^{n} |\langle x, v_i\rangle|^2 \quad \text{for any } x \in V.$$

Further prove that Bessel's Inequality is an equality if and only if $x \in \text{span}(S)$.

(c) Let $V = P_2(R)$, with the inner product

$$\langle f(x), g(x)\rangle = \int_0^1 f(t)g(t)\,dt$$

and with the standard basis $\{1, x, x^2\}$. Use Gram-Scmidth process to obtain an orthonormal basis β of $P_2(R)$. Also, compute the Fourier coefficients of $h(x) = 1 + x$ relative to β.

(d) Find the minimal solution to the following system of linear equations

$x + 2y - z = 1$

$2x + 3y + z = 2$

$4x + 7y - z = 4$

(3+3.5,6.5,6.5,6.5)

6. (a) For the data {(–3, 9), (–2, 6), (0, 2), (1, 1)}, use the least squares approximation to find the best fit with a linear function and compute the error E.

(b) Let T be a linear operator on a finite dimensional inner product space V. Suppose that the characteristic polynomial of T splits. Then prove that there exists an orthonormal basis β for V such that the matrix $[T]_\beta$ is upper triangular.

(c) (i) Let T be a linear operator on $\mathbb{C}^2$ defined by $T(a, b) = (2a + ib, a + 2b)$. Determine whether T is normal, self-adjoint, or neither.

(ii) For $z \in \mathbb{C}$, define $T_z: \mathbb{C} \to \mathbb{C}$ by $T_z(u) = zu$. Characterize those z for which T_z is normal, self adjoint, or unitary.

(d) Let U be a Unitary operator on an inner product space V and let W be a finite dimensional U–invariant subspace of V. Then, prove that

(i) $U(W) = W$

(ii) $W^\perp$ is U-invariant

(6,6,3+3,3+3)